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When do we have information partitions?

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Abstract

This paper axiomatizes in two ways a standard assumption in the economics of information—that an agent's knowledge is representable by an 'information partition' for a state variable. This is done using the 'epistemic models' of Bacharach (1985), set-theoretical structures closely related to deductive systems called 'epistemic logics'. The state variables may be either comprehensive or specific. One set of axioms links partitions with the strong epistemic logic $S5$ and the condition that the state variable be 'epistemically sufficient for itself'. The other formalizes Blackwell's notion of an 'experiment' and permits a much weaker logic. Implications for strategic decision making are discussed.

Introduction

In this paper I axiomatize, in two different ways, a standard working assumption in economics and decision theory—the assumption that a person's knowledge of her environment is representable as an 'information partition'. I do so in terms of set-theoretical structures known as 'epistemic models' which are closely related to deductive systems called 'epistemic logics'.

1. The formal representation of knowledge

1.1 Background

In decision theory, the theory of games, and the economics of information there is burgeoning interest in the axiomatic basis of, and in formal means for representing, the knowledge of rational people. At

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first this new interest centred on 'common knowledge' in a group of people. Today it is increasingly drawn towards knowledge *simpliciter*.

The most powerful single spur to this enquiry has been Robert Aumann's 1976 Agreement Theorem (Aumann 1976), according to which, under rather general conditions, if individuals' opinions are common knowledge among them, these opinions must be the same. Aumann's theorem has excited research because of the near paradoxical nature of certain consequences of it, not least 'no-trade' theorems which deny that, in equilibrium, diverse information can explain exchange in risky assets. In addition, there has been the question of whether two conjectures about the structure of individuals' knowledge which Aumann makes in the course of his discussion of the theorem can be put on a formal footing. To these stimuli we must add mention of the expanding role played by the notion of common knowledge in the theory of games.[1]

Within economic theory the development of axiomatic models of knowledge was initiated by Milgrom (1981), who proposed a set of axioms for a *common* knowledge operator on an algebra of events. Bacharach (1985) sets out an axiomatic theory for knowledge of every order; this is based, like Milgrom's, on algebraic structures (so-called 'epistemic models'), but notes their relationship to the classical epistemic logic $S5$. In some of the most recent work (Gilboa 1986; Kaneko and Nagashima 1988; Samet 1987; Shin 1987) the duality between set-theoretical and deductive modes of analysis has received increasing attention.

There is said to be 'common knowledge between' two persons a and b of a proposition p if all sentences of the form 'i knows that j knows that i knows that ... k knows that p' are true ($i, j, k = a, b$; $i \neq j$); thus the notion is infinitary. The generalization to groups of more than two presents no difficulties. Common knowledge has been in the air since Schelling (1960) drew attention to its role in supporting equilibria in coordination games and other non-cooperative games. In their accounts of language and other conventions, in which common knowledge is pivotal, Lewis (1969) and Schiffer (1972) further defined the notion, of which they were concerned to provide a finite axiomatization.

[1] The idea that in a game players have reciprocal knowledge of the game tree and of each other's rationality has a long history (Milgrom 1981: Schelling 1960): more recently, common knowledge has figured prominently in arguments by backward induction in finitely repeated games (Selten 1978). It is also central in Bayesian accounts of games, but this development is not independent of those springing from Aumann's article.

Structures resembling epistemic models first arose in the *semantics* for deductive theories known as *epistemic logics* (see for example Hughes and Cresswell 1968). These treat the knowledge of rational persons of the propositions[2] of a specified formal language. One way in which an epistemic model can be defined is as an algebraic structure on a set of points; each point is a function ω from the propositions of the formal language to the pair $\{T, F\}$, ('true', 'false') of truth-values; the points ω are interpreted as *possible worlds*. One must put restrictions on possible worlds if they are to merit their epithet. Let k stand for 'a knows that'; then it is only if we insist that $\omega(p) = T$ when $\omega(kp) = T$ that our model reflects an epistemic logic which has the Axiom of Knowledge, as I shall call it, $kp \rightarrow p$. It then 'reflects' the epistemic logic in this sense: it makes the Axiom of Knowledge come out 'true' in every 'possible world'.[3]

1.2 The arrangement of the paper

Three issues in the formal theory of knowledge have been the focus of recent attention;[4] this paper addresses the third of these, the question whether we can justify the Partition Postulate—the assumption that people have 'information partitions'. The next section explains why the Partition Postulate is an issue. It begins by describing the first two issues, which are the historical source of the third; next, it gives examples of different species of 'information patterns', of which information partitions are only one; describes the traditional role of the Partition Postulate in economics and decision theory; and notes the association between information partitions and a certain epistemic logic ($S5$) which is often thought objectionably strong. Section 3

[2] This is imprecise: languages consist of sentences, and epistemic logics deal with logical relationships among *sentences* rather than the propositions which they express; the distinction is, however, unimportant for the purposes of this paper.

[3] Epistemic logic as an independent branch of modal logic stems from Hintikka (1962) who, by revealing the 'depth logic' underlying ordinary usage, resolved old puzzles about knowledge, such as what makes it paradoxical to say 'p and I believe that not p'. But epistemic logic has thrown up its own paradoxes, such as that of the Knower (Anderson 1983): these have resurfaced in recent claims (Samet 1987: Shin 1987) that apparently reasonable axioms about knowledge imply the knowledge of falsehoods.

[4] Other current issues in the formal theory of knowledge are not directly relevant to the question I address. They include: (i) doubts about the computability of the knowledge which epistemic logics ascribe to knowers (Shin 1987), doubts which are related to the epistemic paradoxes; (ii) the relation between the logic of knowledge and that of mere rational beliefs (which can be mistaken) (see Hintikka 1962: Samet 1987). Of potential importance for economics and decision theory is the question of what to take as the objects of knowledge (events, propositions, sentences), and the consequences of their 'intensionality' (see Bacharach 1985: Kaneko and Nagashima 1988).

introduces the main results of the paper, two sets of conditions for people to have information partitions. Sections 4 to 7 contain the formal analysis. The final section considers the bearing of the results on how we should interpret players' information partitions in games.

2. The partition postulate

2.1 Aumann's first conjecture and the familiarity postulate

In this paper the only non-epistemic propositions I consider are of the form $s \in S$; s will be called a 'state variable', and its values 'states'.[5] For any state variable s, I shall be interested in persons' 'information patterns' for s, meaning by this what they know in different states about which state obtains. The set of states a person does not exclude (that is, thinks could possibly obtain) is called her 'fix' on the state. We may define 'information pattern' with somewhat more precision in terms of this notion: a person's information pattern with respect to s is a set of ordered pairs $\langle s, S\rangle$ ($s \in S$, $S \subseteq \Sigma$); $\langle s, S\rangle$ is in her information pattern just if there are circumstances in which the state is s and her fix on the state is S.

An 'information partition' (i.p.) is a special kind of information pattern. A person has the i.p. P for s if P is a partition of Σ and, for all s in Σ, whenever the state is s her fix on the state is $P(s)$, the member of P containing s. Aumann's Agreement Theorem concerns two people each of whom is assumed to have an i.p. for a state variable s. Aumann's First Conjecture is a characterization of common knowledge in terms of these personal i.p.'s, namely: in state s the event that p is common knowledge between the two people obtains just if it includes $P^*(s)$, where P^* is the meet of the two i.p.'s; Bacharach (1985) gives formal expression to this conjecture and proves it by the use of epistemic models.

Aumann's Second Conjecture is that there is necessarily common knowledge among people *of* their personal i.p.'s for s if s is a *global* state variable. This is the conjunction of two independent claims: (a) people have common knowledge of each other's information *patterns* for s (I shall call this the Familiarity Postulate); (b) information patterns are i.p.'s (the Partition Postulate). The essence of Aumann's argument for the Familiarity Postulate (1976, 1987) is that since s specifies how all things are, it specifies *inter alia* how epistemic things are. But Aumann adds the disclaimer that this cannot be shown within the formal model; this is capable of representing the knowledge

[5] Since any collection of propositions can be translated into these terms, we lose no generality by this device.

constituting information patterns, but not in addition knowledge *about* this representation.

The reminder that formal models cannot always 'talk about themselves' is salutary. However, Aumann is unduly pessimistic: versions of the Familiarity Postulate *can* be shown formally. It is proved in Bacharach (1985) for *atomic* epistemic models, which are appropriate whenever there is a finest way in which the theorist is interested in discriminating states of affairs.[6] The proof rests on two properties of epistemic models, which I shall also make use of here. Call a proposition 'familiar' which is true in *every* possible world ω of an epistemic model. The first property is that if p is familiar then so is the proposition that i knows that p; then by induction such a p is *common* knowledge in all ω. Familiarity breeds common knowledge. The second property is that information patterns are representable in epistemic models for any state variable s whose states are representable in them.

2.2 Information patterns

The form of a person's information pattern for a state variable—in particular, whether or not it is an i.p.—depends crucially on two things: the strength of her rationality, and the globality of the state variable. Some simple examples illustrate. In Figs. 1 to 4 the points represent states, the closed curves enclose fixes, and the arrows lead from states to the fixes which can occur in them. Call the person a. In Figs. 1 to 3 a's information pattern is a function; the fix on s is determined by s alone. In this case I shall also say that s is 'epistemically sufficient for itself' (e.s.i.).

Let $\Sigma = \{1, 2, 3, 4\}$. In Fig. 1, a has an information partition. In Fig. 2, the fixes cross over. However, the following argument suggests this pattern may not be possible. Suppose $s = 1$, then a can reason: 'if s were 3, I would not know that s wasn't 4. But I do. So $s \neq 3$.' Thus she does not allow the possibility $s = 3$, contrary to the assumed information pattern. But notice that this argument appeals to two further premisses: (a) a knows the fixes she would have in other states; (b) she satisfies the requirement that if one knows a certain thing one knows that one does; I shall call this requirement on someone's knowledge the Axiom of Transparency. It is by no means obvious that these additional premisses must be true; nor, then, that Fig. 2 is

[6] Bacharach assumes the Partition Postulate, but it plays no essential role in the result. Recently, Shin (1987) has generalized Bacharach's result to non-atomic epistemic models.

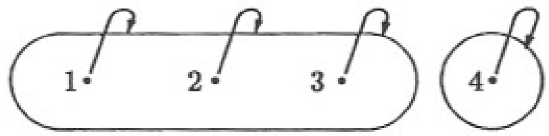

Figure 1

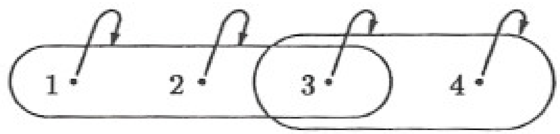

Figure 2

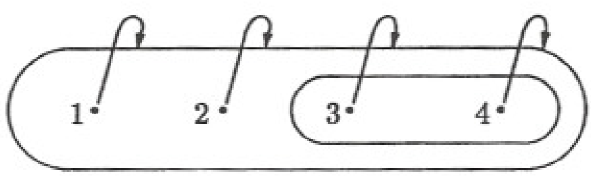

Figure 3

impossible. In Fig. 3 the fixes form a topology, and the above argument does not apply; but this configuration also leads to contradiction if we accept (a) and that (c) a satisfies the requirement that if one does *not* know a certain thing then one knows that one does not, which I shall call Axiom of Wisdom. If $s = 1$, a reasons: 'if s were 3, I would know that $s \neq 1$ and $s \neq 2$. But I don't. So $s \neq 3$.' The Axioms of Transparency and Wisdom may be regarded as ***constraints of rationality.*** These examples suggest that the Partition Postulate may fail in the absence of rationality constraints.

In Fig. 4a, $\Sigma = \{1,2\}$. The state variable is not e.s.i., and *a fortiori* a does not have an i.p. Failure of s to be e.s.i. implies that there is another state variable which affects a's fix on s, and so that s is not a global state. The pattern in the figure could arise if, for example, a had an i.p. for the state variable $\langle s,t \rangle$ as in Fig. 4b (where 11 denotes $\langle 1,1 \rangle$, etc.). This illustrates that the property of partitionality of an information pattern does not project. Nor does it 'project outwards': an i.p. for s does not imply one for $\langle s,t \rangle$. I note that the pattern in this example is possible even if a satisfies (a) and both the Axiom of Transparency and the Axiom of Wisdom.

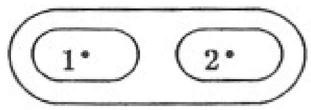

Figure 4a

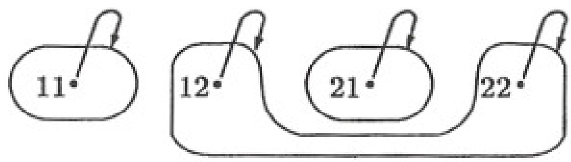

Figure 4b

2.3 The use of information patterns

Information patterns figure prominently in economics and decision theory, but their assumption has seldom been defended, and never in a fully formal way. They have been used to model, among other things: (a) players' knowledge of the current position in the course of an extensive-form game (Von Neumann and Morgenstern 1944); (b) the possible future information-states of a Bayesian decision maker who is choosing a decision rule (Radner 1961), for example of an individual consumer or producer in an economy with markets in state-contingent commodities (Radner 1968); (c) the differential private information of agents who draw inferences from each other's public behaviour, as in the above-mentioned work of Aumann.

I shall comment on the role of i.p.'s in games in §8. They arise as a model of (b) in the following way. The decision maker (a, say) will presently observe a 'signal' y given by a function η (the 'information function') of a variable s over a set Σ; this will put her into an information state in which her fix on s is S_y, the equivalence class of y modulo η. The proofs of epistemic commonplaces are often non-trivial, and so it is, we shall find, with this intuitively obvious claim. One reason is that the intuitive conclusion that her fix is S_y depends on the idea that *all she knows about s* is the value of the signal function, and this idea is not easy to pin down rigorously. Be that as it may, the claim implies that a has an i.p. for s, since the S_y partition Σ. I shall call situations of the form just described 'experiments', following Blackwell (1953). The model has proved extremely tractable and fruitful.[7]

[7]Situations in which the signal is 'noisy', that is, a random function of the

Suppose a is a Bayesian decision maker with a probability measure on Σ, and that after observing the signal she must do an act whose payoff depends on s; it is a familiar thesis that she may proceed by choosing in advance a 'decision function' or 'strategy' δ, a function of her future fix S_y, since her expected utility now from choosing δ optimally is equal to her expected utility now from the observe-first-and-decide-after policy.[8] Now in order to evaluate a strategy δ in which she will do $\delta(s)$ if her fix is S, she must evaluate the probability that her fix will be S, and in order to do this she must identify a set of states in which she will have it. Thus the rationale of strategic decision making depends on there being a function from s to her fix on s, that is, on s being e.s.i. That there is such a function is ensured by her having an i.p.[9]

In some theories in which i.p.'s have been used, weaker assumptions will do the job. For example, for the Agreement Theorem it has been shown that the information patterns need only be topologies (Kaneko 1987; Samet 1987; Shin 1987). But such redundancy findings are unlikely in other applications. The question will not be pursued in the present paper, whose purpose is to investigate the conditions of existence of information partitions, not their dispensability.

2.4 Partitionality and strength of knowledge

An equivalence has long been known between two conditions on a person's knowledge: that it obeys a certain epistemic logic, $S5$, and that it is partitional in a certain way; I shall call this equivalence the Partition Theorem. A helpful, if strictly improper, statement of the partitionality condition in question is that, as ω ranges over the set Ω of all possible worlds, the person's fixes on ω partition Ω.[10] The same idea can be rigorously expressed in terms of an 'atomic state variable'

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state, can be subsumed under the above formulation; the ranking of information functions by 'informativeness' (the utility in varying circumstances of the information they convey), is identical with that according to the relative fineness of the corresponding i.p.'s; and so forth.

[8] She is modelled as deciding to proceed thus: choose now ($t = 1$) a function δ of y and at $t = 2$ if she observes y do act $\delta(y)$. If she chooses δ her utility payoff will be $u(s, a) = u(s, \delta(\eta(s)))$. At $t = 1$ she has a subjective probability function on Σ, and she chooses δ to maximize $Eu(s, \delta(\eta(s)))$.

[9] Furthermore, the decision maker must know this 'fix function'. We shall see in §5 that this is guaranteed if she has an i.p.

[10] Improper because possible worlds are not states, and the notion of fix is not defined over them. Furthermore, the set of propositions true in a possible world is at least countably infinite, and so is not in general expressible in a sentence. This creates a difficulty since formal theories of knowledge express the contents of knowledge in sentences.

s rather than ω.[11] In either formulation the variable for which the person has an i.p. is a complete specification of relevant features of the world. The logic $S5$ contains among its axioms both the Axiom of Transparency and of Wisdom; we glimpsed earlier, in the examples of §2.2, the potential of these axioms to force partitionality.

Unfortunately for the project of justifying the Partition Postulate, the epistemic logic $S5$ appears very, perhaps excessively, strong. The Axiom of Wisdom, in particular, has to many seemed questionable even for ideally rational agents (Bacharach 1985; Eberle 1974; Hintikka 1962; Rescher 1984). Binmore and Brandenburger (1988) note that it is equivalent to the implausibly exigent 'if a doesn't know she doesn't know something, then she knows it'. If we take knowledge to be 'active', so that knowing that p entails having a thought with the content p, then unawareness of issues—the mere absence of issues from one's epistemic agenda—gives a host of violations of the Axiom of Wisdom; but such epistemic gaps are commonplace, and moreover they give rise to informational asymmetries in strategic interactions that may be crucial.[12] Another rich seam of violations is that a person may be falsely convinced of something.[13]

3. Two bases for information partitions

Let us take stock. On the one hand, i.p.'s are a fundamental tool in the analysis of agents' knowledge in decision theory and economics; on the other, the Partition Theorem suggests that to be entitled to use them we may have to accept a false picture of epistemic rationality which is too high a price to pay for them. Things, however, are not as bleak for the defence of i.p.'s as this picture paints them, for the connection between the i.p.'s we are interested in procuring and $S5$ is less strict than appears at first sight. I.p.'s are needed for certain *modelling* purposes, and two basic features of the typical modelling problems lessen the relevance of the Partition Theorem. These are that it may be appropriate to assume 'endowments' of knowledge exogenously, and

[11] A version of this theorem is given in Bacharach (1985).

[12] Milgrom and Roberts (1986) note that a seller may withhold from a potential buyer his private information about some dimension of his product because 'bringing up' the question may be damaging (even when the information is favourable it may be, by arousing fears where none were present). The seller is coy because he suspects the buyer is ignorant of her own ignorance about the dimension because she has not 'thought of it'.

[13] If I am convinced that p then, trivially, I am convinced that I have a true belief that p; and at least some varieties of conviction carry with them the belief that they are justified. In that case I believe that I have a justified true belief that p and so cannot know that I do not know it. Yet I do not know it, for p is false.

that the state variable of interest may be 'topical' rather than global. The two main results of this paper (Theorems 1 and 2) give conditions for i.p.'s which lean heavily on these two features respectively, and at the cost of some over-simplification might be called the 'endowment theorem' and the 'topicality theorem'.

$S5$ and the other classical systems of epistemic logic are silent on the empirical sources of people's knowledge of the world. (In this they resemble subjectivist models of agents' beliefs which merely impose a coherent (probabilistic) structure on these beliefs.) The concrete situations in which people find themselves are partly characterized by epistemic 'endowments'. On one, 'situationist', methodology, it is not the job of social scientists to trace the behaviour of agents back to the ultimate sources of human knowledge, but rather to show how it depends upon these more or less 'near-in' sources of it; where *these* come from is exogenous to the enquiry in hand. This contrasts with the globalist methodology of the Neo-Bayesians. Most of economic theory, even general equilibrium theory, is situationist in method and assumes epistemic endowments exogenously. For example, it is constitutive of being in an extensive-form game that you receive, when it is your turn to move, one or another 'umpire's report'; and the role of informational endowments is explicit in economic theories in which agents receive market signals or face other experiments in the sense of §2.3.

What agents know consists of these endowments plus their inferences from them. Endowments and inference are alternative ways of acquiring knowledge: like the owner of a monetary fortune, the possessor of an i.p. may equally well have started out poor and gifted, or rich and dull. Theorem 1 exemplifies this truth. It is shown that if a person's epistemic endowment is of a certain form, she 'ends up with' an i.p. though only satisfying the epistemic logic known as T; T is substantially weaker than $S5$, lacking the Axioms of Transparency and Wisdom. The endowment in question is that of the signal-receiver in the 'experiment' model as we intuitively interpret that model; Theorem 1 may be regarded as a formal demonstration of the 'epistemic commonplace' that the signal-receiver in an experiment has the information partition induced by the information function.

The Partition Theorem relates to a global state variable, which specifies how everything is, but the question this paper addresses is for *any* state variable, global or otherwise. We must therefore ask whether the Partition Theorem generalizes to an arbitrary state variable s. We find that it does not do so unconditionally: for arbitrary s, $S5$ neither implies nor is implied by an i.p. for s. Theorem 2 provides the appropriate generalization and, thereby, a 'strong knowledge' basis for the Partition Postulate for an arbitrary state variable. Say that someone's knowledge is 'strong w.r.t. s' if it obeys the Axioms of Transparency and Wisdom restricted to propositions of the form $s \in S$. Theorem 2 says that strong knowledge w.r.t. s is a sufficient condition for an i.p. for s *if* s is e.s.i. This raises the question of when variables will be e.s.i., which I discuss briefly in §7. It also provides some further comfort for users of i.p.'s, since the strong axioms are the less strong the more specific are the state variables for which they are adopted.

4. Epistemic models

Definition. Let I be a finite set (of *persons*). An *epistemic model* (e.m.) is a pair $(\mathcal{F}, K)$, where $\mathcal{F}$ is an atomic sigma-field of sets, with universal set Ω, and K is a set of operators $\{K_i : i \in I\}$ on $\mathcal{F}$.

Interpretation. A point ω of Ω is interpreted as a 'possible world' (p.w.), a specification of a state of affairs obtained by evaluating each proposition of some reference class in which we are interested as either true or false; if E is the set of p.w.'s in which a given proposition p of this class is valued true, it is called the *truth-set of* p or the *event that* p, and written $|p|$. If E is the event that p, K_iE is interpreted as the *event that i knows that p*. Further, if for some event E^*

$$E^* = \bigcap K_{i_1} \dots K_{i_n} E,$$

where the intersection is over all strings $\langle i_1, \dots, i_n \rangle$ of elements of I, E^* is interpreted as the *event that there is common knowledge among I that p.*

The interpretation I have just given stands up only if possible worlds are subjected to a set of restrictions corresponding to the meanings of the logical connectives $\wedge$ (and), $\neg$ (not), $\rightarrow$ (if...then...), etc. figuring in complex propositions. These restrictions are equivalent to restrictions on the mapping $|\cdot|$ of propositions into events. Thus we insist that $|p \wedge q| = |p| \cap |q|$, $|\neg p| = -|p|$, $|p \rightarrow q| = -|p| \cup |q|$ (where '$-$' denotes complement in Ω), and so forth. I henceforth assume this to be so. Similarly, the meaning of the epistemic connective k_i (i knows that) forces a set of restrictions on the mapping into events of propositions which contain 'i knows that'. In this case, since we may understand this connective in various ways, there may be several non-equivalent plausible sets of restrictions. Each such restriction is in effect an axiom which partially defines the notion of knowledge. I next define one plausible set of restrictions of this sort, with which I shall work in most of the paper; the labels beside the restrictions are those of the axioms, etc. to which they correspond. E, F range over $\mathcal{F}$, and i ranges over I.

(A1)	$K_iE \subseteq E$	(*Axiom of Knowledge*)
(A2)	$K_i(E \cap F) = K_iE \cap K_iF$	(*Conjunction Axiom*)
(A3)	$K_i\Omega = \Omega$	(*Rule of Epistemization*).

Interpretation. (A1) is equivalent to the restriction $|k_ip| \subseteq |p|$ on the proposition-event mapping. This restriction means that in any possible world in which it is true that i knows that p, p is itself true; thus it expresses the conceptual truth that knowledge is factive, the Axiom of Knowledge. In the same way (A2) expresses the axiom that someone knows the conjunction of two things just if she knows each of them. (A3) says, *grosso modo*, that a person necessarily knows that which is necessarily so, and does the work of the Rule of Epistemization in epistemic logic.

I shall denote by $\mathfrak{M}0$ *the class of all e.m.'s which satisfy* (A1)–(A3). I shall use the following simple properties of $\mathfrak{M}0$, whose proofs I omit.

Proposition 1. (a) *If* $E \subseteq F$ *then* $KE \subseteq KF$; (b) *if* $E \subseteq F$ *then* $-K{-E} \subseteq -K{-F}$; (c) *if* $K(E \cup F) = \Omega$ *then* $K{-E} \subseteq KF$.

The *logic* T is a weak epistemic logic which lacks the Axioms of Transparency and Wisdom. It is obtained by adjoining to propositional calculus the Axiom of Knowledge, the Conjunction Axiom, and the Rule of Epistemization; there is, predictably, an intimate connection between T and the class $\mathfrak{M}0$. An e.m. is said to *verify* the proposition p if $|p| = \Omega$ in it. It can be shown that p is verified by all e.m.'s of $\mathfrak{M}0$ if and only if p is an axiom or theorem of the epistemic logic T.[14] This connection legitimizes the use of $\mathfrak{M}0$ to establish results about the reference class of propositions. Showing that p is verified throughout $\mathfrak{M}0$ amounts to a proof of p (for our success means that it *can* be proved in T), while displaying an e.m. of $\mathfrak{M}0$ in which p is not verified permits the conclusion that $\neg p$ is a logical possibility (for $\neg p$ is consistent in T).[15] This generalizes usefully: if $\mathfrak{M}0_p$ is the subclass of e.m.'s of $\mathfrak{M}0$ which verify the proposition p, then showing that q is verified throughout $\mathfrak{M}0_p$ shows that q follows

[14] See for example Hughes and Cresswell (1968, Ch. 17, Theorems 1 and 2 and *passim*).

[15] For instance, in this way we can prove $k_i(p \to q) \to (k_ip \to k_iq)$, and we can show that it is possible that $k_ip \wedge \neg k_ik_ip$.

(in T) from p, and displaying an e.m. of $\mathfrak{M}0_p$ in which q is not verified shows that $\neg q$ is consistent with p (in T).

Definition. If Σ is a set $\{s_k : k = 1, \ldots, n\}$ I shall say that s is *a state variable over* Σ if exactly one proposition of the form $s = s_k$ is true.[16] If s is a state variable over Σ, for all $s' \in \Sigma$, $S \subseteq \Sigma$, $|s'|$ will denote the event that $s = s'$ and $|S|$ the event that $s \in S$. I shall write v for the proposition that s is a state variable over Σ. Then the *event that s is a state variable over* Σ is the set

$$|v| = \bigcup_{s \in \Sigma} \left[|s| \cap \bigcap_{s' \neq s} - |s'| \right].$$

Let us now introduce a further restriction on the mapping $|\cdot|$, viz.

$$|v| = \Omega. \tag{1}$$

I note that (1) is met just if $\{|s| : s \in \Sigma\}$ is a partition of Ω.

I write $\mathfrak{M}$ for $\mathfrak{M}0_v$, the class of e.m.'s which satisfy both (A1)–(A3) and (1); if p is any proposition, let $\mathfrak{M}_p$ be the subclass of e.m.'s of $\mathfrak{M}$ which in addition verify p. Just as $\mathfrak{M}0$ represents the epistemic logic T, so $\mathfrak{M}$ represents the formal theory set in T whose sole non-logical axiom is v.

It is worth pausing to consider the interpretation of a single e.m. of $\mathfrak{M}$. It collects a set of alternative possible states of affairs in which the axioms of the underlying theory, including v, are true. But not all such states of affairs; hence an e.m. may verify propositions which are not entailed by the theory. For example, some e.m.'s of $\mathfrak{M}$ verify that s has two possible values, others the Axiom of Transparency, and so on. The variation in states of affairs that a single e.m. encompasses is needed to represent epistemic phenomena, for these involve persons' having attitudes towards a range of possibilities. A particular e.m. defines an 'epistemic set-up', that is, a collection of states of affairs in each of which a person has such and such epistemic attitudes towards the others. Whereas in a single possible world in an e.m. it is either true or false that a person knows that $s \in S$, an e.m. shows the way her knowledge of s varies as circumstances vary; thus it can represent what Von Neumann and Morgenstern call a person's 'knowledge capacity'. An information partition is such a capacity.

From now on I shall be mainly concerned with the knowledge of a single individual, a; for neatness I shall write K for K_a, k for k_a, and so forth.

[16] The restriction to finiteness is made chiefly to avoid the need to appeal to epistemic logics which treat knowledge of propositions which quantify over infinite domains.

5. Information patterns in $\mathfrak{M}$

Definitions. For any e.m. of $\mathfrak{M}$, I define the following events. The *event that a has fix S on s* is

$$|\text{fix}\, S| = \bigcap_{-S}(K - |s|) \cap \bigcap_{S}(-K - |s|) \tag{2}$$

(where $-S$ denotes the complement of S in Σ). If ϕ is a function from Σ to $\wp(\Sigma)$, the *event that a has the fix function ϕ for s* is

$$|\text{f f}\phi| = \bigcap_{\Sigma}[-|s| \cup |\text{fix}\, \phi(s)|]\,. \tag{3}$$

I note that $|\text{f f}\phi| = \Omega$ or $\emptyset$. Thirdly, the *event that a has the information partition P for s* is

$$|\text{ip}P| = |\text{f f}\phi| \tag{4}$$

where $P = \{\phi(s) : s \in \Sigma\}$ and P is a partition of Σ.

Interpretation. The event $|\text{fix}\, S|$ is the set of possible worlds in which the following is true: a rules out all and only the values of s outside S (equivalently, she 'allows' all and only values of s in S); that is, a's 'information set' for the variable s or, in a terminology which is gaining currency, a's 'fix on s', is S. The event in (3) is the event that a 'has the fix function ϕ' in this sense: for all s' in Σ, if $s = s'$ then a's fix on s is $\phi(s')$. Finally, the event in (4) is the event that a 'has an information partition' in this sense: she has a fix function ϕ for which the $\phi(s)$ make up the partition P of Σ. Thus to have an i.p. is to have a special kind of fix function.

Although the definition of fix does not require the fix to contain the true value of s, it must. For we can show that $|-\text{fix}(S)| \cup |S| = \Omega$, or, equivalently

Proposition 2. $\quad |\text{fix}(S)| \subseteq |S|$.

Proof. By (2) and (A1), $|\text{fix}(S)| \subseteq \bigcap_{-S}(K - |s|) \subseteq \bigcap_{-S}(-|s|) = |S|$. ∎

Thus if a person has an i.p. P, then if the state is s her fix is $P(s)$, the member of P which contains s, in accordance with the usual meaning of 'information partition'.

There is no reason why a person cannot have a certain fix function, according to an e.m., in some circumstances and not in others; no reason, that is, why the event in (3) must be either $\emptyset$ or Ω. Specializing, someone may have a certain i.p. only in certain circumstances. To say that the fix a person has on s may vary with s is simply to say that at each value of s her fix may depend on other factors besides s; for instance, how my fix on the score in the (cricket) test match varies with the score depends on whether or not I can get the ball-by-ball commentary on my radio. Consider the example of Fig. 4b. It may be represented by an e.m. of $\mathfrak{M}$, μ_1, say, which has events for values of t as well as of s. Let ϕ_1, ϕ_2 be the two fix functions on Σ defined by $\phi_1(1) = \{1\}$, $\phi_1(2) = \{2\}$, $\phi_2(1) = \phi_2(2) = \{1, 2\}$; then the events that a has the fix functions ϕ_1 and ϕ_2 are the events $|t = 1|$ and $|t = 2|$ respectively. I remark that the existence of μ_1 permits the conclusion, for the reasons mentioned earlier, that it is not a theorem (in the logic of T) that a's fix on s is independent of t.

On the other hand, an e.m. may *verify* that a has the fix function ϕ; it does so if, in it, $|s| \subseteq |\text{fix}\,\phi(s)|$ for all $s \in \Sigma$. If the e.m. μ verifies that a has the fix function ϕ (or the i.p. P), I shall say that *a has the fix function ϕ (resp. i.p. P) in μ*. Similarly, I shall say that *there is common knowledge among I that p in μ* if μ verifies the event that there is common knowledge among I that p.

Let $\mathcal{S}$ denote the subfield of $\mathcal{F}$ consisting of *s-events*, that is, events of the form $|S|$, $S \subseteq \Sigma$. I shall say that *a's knowledge is closed w.r.t. s in μ* (alternatively, *s is epistemically sufficient for itself (e.s.i.) in μ*) if $\mathcal{S}$ is closed under K in μ. I shall use the following two points later.

Proposition 3. *The person a's knowledge is closed w.r.t. s in μ if and only if a has a fix function in μ.*

Proof. (a) (only if). For any $S \subseteq \Sigma$, $|\text{fix}\,S|$ is, by (2), an intersection of events of the forms $K|S'|$ and $-K|S''|$ (S', S'' in Σ). By closedness these are *s-events*, hence $|\text{fix}\,S| = f(S)$ for some subset $f(S)$ of Σ. Set $\phi(s) = S$ if and only if $s \in f(S)$; then $|\text{f}\,\text{f}\phi| = \Omega$. (b) (if). If a has fix function ϕ in μ then, for all s, $K - |s| = -|\phi(s)|$, an *s-event*; hence for any $S \subseteq \Sigma$, by several applications of (A2), $K|S| = \bigcap_{-S} K - |s|$ is an *s-event.* ■

Proposition 4. *If a has the fix function ϕ in μ then in μ there is common knowledge among I that a has the fix function ϕ.*

Proof. $K_{i_1} \ldots K_{i_n} |\text{f}\,\text{f}\phi| = \Omega$ for every string $(i_1, \ldots, i_n)$ of elements of I, by (A3). ■

6. The simple experiment basis for information partitions

Definition: For any two fix functions ϕ_1, ϕ_2 if

$$\phi_1(s) \subseteq \phi_2(s) \text{ for all } s \in \Sigma$$

say that ϕ_1 is *as sharp as* ϕ_2, and write $\phi_1 \succeq \phi_2$. The binary relation $\succeq$ partially orders the fix functions. In virtue of this definition, a has a fix function as sharp as ϕ just if, for all s in Σ, if the state is s she rules out all states outside $\phi(s)$; thus the set

$$|\mathrm{ff} \succeq \phi| = \bigcap_{\Sigma}[-|s| \cup \bigcap_{-\phi(s)}(K - |s'|)]$$

is *the event that a has a fix function for s as sharp as* ϕ.

Definition: Call an e.m. μ *canonical given p*, if $\mu \in \mathfrak{M}_p$ and there is a fix function ϕ such that

$$|\mathrm{ff}\phi| = \Omega \text{ in } \mu \tag{5}$$

$$|\mathrm{ff} \succeq \phi| = \Omega \text{ in all e.m.'s of } \mathfrak{M}_p. \tag{6}$$

Write Φ_0 for the set of fix functions which satisfy (6). Then

Proposition 5. *If ϕ satisfies (5) and (6), ϕ is maximal w.r.t. $\succeq$ in Φ_0.*

Proof. If $\phi' \in \Phi_0$, by (5) there is an e.m. in which $|\mathrm{ff} \succeq \phi'| \cap |\mathrm{ff}\phi| \neq \emptyset$ and in which therefore the event that $\phi \succeq \phi'$ is non-null, whence $\phi \succeq \phi'$; since $\phi \in \Phi_0$, it is maximal in Φ_0. ∎

Interpretation. We have seen that verification throughout $\mathfrak{M}_p$ is equivalent to derivability in T from p. It follows from Proposition 5 that if μ is a canonical e.m. p, then in μ, a has the sharpest fix function that is guaranteed by the assumption p (in the logic of T). In other words, a has in the epistemic set-up described by μ the knowledge (capacity) concerning s that is a consequence of p (in T), and no more. An immediate corollary is that, given p, in order to explain a's having a fix function strictly sharper than ϕ, we must make *further assumptions* about her knowledge. Thus ϕ is what we must conclude she knows if she is a subject of T and if *all we assume her to know* is what is given in p.

I next define a proposition, e, which expresses in a formal way that a is the observer in an experiment. The main result of this section is that in an e.m. which is canonical given e, a has an i.p. Let us specify P to be the partition of Σ

$$P = \{S_1, \ldots, S_m\} \tag{7}$$

where

$$S_j = \{s_{j1}, \ldots, s_{jn(j)}\} \quad (j = 1, \ldots, m).$$

Let $Y = \{y_1, \ldots, y_m\}$ be a set (of ***observations***), and η a function from Σ to Y such that

$$\eta \text{ induces the partition } P \text{ of } \Sigma.$$

By *a **knows*** η let us mean that a knows, for each state s in Σ, that if s transpires then y takes the value $\eta(s)$ and takes no other, and by *a **receives** y* let us mean that, for all observations y in Y, if y transpires then a knows that it does. Finally, let e be the following proposition:

(e) s is a state variable over Σ, a knows η, and a receives y.

Consider the events

$$\begin{aligned} E_1 &= |v| \\ E_2 &= K\left[\bigcap\nolimits_{\Sigma}[-|s| \cup |y =_1 \eta(s)|]\right] \\ E_3 &= \bigcap\nolimits_{y' \in Y}[-|y = y'| \cup K|y = y'|] \end{aligned} \tag{8}$$

where $|y =_1 y'|$ is the event that y takes the value y' and no other defined by

$$|y =_1 y'| = \left[|y = y'| \cap \bigcap\nolimits_{y'' \neq y'}[-|y = y''|]\right]. \tag{9}$$

Interpretation. E_1, E_2, E_3 are respectively the events that s is a state variable over Σ, that a knows η, and that a receives y. Thus the event that e is the set

$$|e| = E_1 \cap E_2 \cap E_3 .$$

We may also describe $|e|$ as the event that a is the observer in an experiment with information function η. The main result of this section is

Theorem 1. *If μ is canonical given e then a has the i.p. P in μ.*

Proof of Theorem 1. (a) We show first that $|\text{f f} \succeq \phi| = \Omega$ in every e.m. of $\mathfrak{M}_e$, where ϕ is the fix function agreeing with P. For this it suffices to show that in every such e.m., for all s, s', if $s' \notin P(s)$ then $|s| \subseteq K-|s'|$. Write $y = \eta(s)$, $y' = \eta(s')$, and write $|y|$, $|y'|$ for the events that the observation is y, y' respectively. Consider any e.m. of $\mathfrak{M}_e$. Since $E_2 = \Omega$, by (A1) the argument of K in (8) is Ω, whence

$|s| \subseteq |y|$ since $(-|s| \cup |y|)$ is a factor of this argument. But $|y| \subseteq K|y|$ since $E_3 = \Omega$, so

$$|s| \subseteq K|y|. \tag{10}$$

Since $s' \notin P(s)$, $y \neq y'$. The event $(-|s'| \cup |y =_1 \eta(s')|) = \Omega$ as it is also a factor of the argument of K in (8); hence $(-|s'| \cup -|y|) = \Omega$ by (9), $K(-|s'| \cup -|y|) = \Omega$ by (A3), and $K|y| \subseteq K{-}|s'|$ by Proposition 1. Thus $|s| \subseteq K{-}|s'|$ by (10).

(b) Secondly, we display an e.m. μ of $\mathfrak{M}$ in which $E_1 = E_2 = E_3 = |\text{ip}\, P| = \Omega$ and which is therefore an e.m. of $\mathfrak{M}_e$ in which $|\text{ip}\, P| = \Omega$, and so, in view of (a), canonical given e. It follows at once that, as the theorem asserts, $|\text{ip}\, P| = \Omega$ in every e.m. canonical given e. Let Ω be the union of the disjoint sets Ω_j $(j = 1, \ldots, m)$ where Ω_j is the set $\{j1, \ldots, jn(j)\}$. For $j = 1, \ldots, m$, set

$$|s_{jr}| = \{jr\} \quad (r = 1, \ldots, n(j)) \tag{11}$$

$$|y_j| = \Omega_j \tag{12}$$

$$K|S_j| = \Omega_j \tag{13}$$

$$K{-}|s_{jr}| = -\Omega_j \quad (r = 1, \ldots, n(j)). \tag{14}$$

This specifies μ.[17] It is easy to see that μ verifies E_1, by (11). From (11), $|S_j| = \Omega_j$, whence by (12) the event that, for all j, if $s \in S_j$ then y has the value $\eta(s)$ and no other, is Ω; provided $\mu 1$ is in $\mathfrak{M}$, the event E_2 that a knows this is also Ω, by (A3). From (12) and (13) $K|y_j| = |y_j|$ for all j, confirming that $E_3 = \Omega$. By (4), $|\text{ip}\, P| = \Omega$ if, for each j, and for each s in S_j, (i) $|s| \subseteq K{-}|s_{kr}|$ for all $k \neq j$ and (ii) $|s| \subseteq -K{-}|s_{jr}|$ for $r = 1, \ldots, n(j)$. But $|s| \subseteq \Omega_j$ by (11); (ii) follows by (14), and $|s| \subseteq \Omega_j \subseteq K|S_j| \subseteq K{-}|s_{kr}|$ by (13) and Proposition 1 since $|S_j| \subseteq -|s_{kr}|$ by (11). It remains to check that $\mu 1$ is in $\mathfrak{M}$. It is clear that the non-epistemic events given in (11) and (12) conform to the assignment rules. It is known (Hughes and Cresswell 1968) that the operator K does so if there is a reflexive binary relation R on Ω such that, for all events E in $\mathcal{F}$,

$$KE = \{\omega \in \Omega : R(\omega) \subseteq E\}.$$

These requirements are met by the relation R defined by $R(jr) = \Omega_j$. ∎

[17] But for the other Ks. The proof goes through without difficulty if, for example, we make all but a ignoramuses.

Remarks on Theorem 1. Say that a 'conducts the simple experiment (e, T)' if she has just the knowledge yielded by e in T; that is, if she knows as much about s as, and no more than, follows in T from the assumption e. Construing 'knows as much about s' as the binary relation $\succeq$, Theorem 1 says that a person who conducts the simple experiment (e, T) has the i.p. P. The theorem thus gives precise form to the old intuition that someone in an experiment has the i.p. induced by its information function. It makes precise, among other things, the intuitive idea that this is so because such a person, *to begin with*, knows *only* the information function and the signal. The notion of her 'knowing to begin with' is formalized by asking what follows from the assumption e. The notion that she knows only this to begin with is captured by the maximality of the i.p. with respect to $\succeq$, and the consequence of this maximality pointed out above.

It is clear that if a were, in real time, provided with the knowledge specified in e, she would be able to arrive at the knowledge in P by a sequence of real-time inferences. This is because the axioms of T that yield for us the conclusion that she has the i.p. P are conditionals in which the antecedent gives an item of knowledge and the consequent knowledge of something deducible. However, epistemic logics such as T, and likewise the corresponding systems of epistemic models such as $\mathfrak{M}$, do not treat the person's deductions explicitly. (For formal epistemic theories that do, see Eberle 1974; Kaneko and Nagashima 1988).

7. The strong knowledge basis for information partitions

In this section I give the second explanation of i.p.'s, in terms of 'strong' knowledge. In it, having an i.p. with respect to s depends on having knowledge that is strong—that is, satisfies the Axioms of Transparency and Wisdom—at least where s-events are concerned. But this is not enough by itself. I show that for i.p.'s to follow, knowledge must be closed w.r.t. s as well as strong for s. This prompts the question why and when someone's knowledge should be closed w.r.t. a variable. One direction this question leads us is back to simple experiments.

Definition. If μ is an e.m. of $\mathfrak{M}$, I shall say that *a's knowledge is strong w.r.t. s in μ* for all events E in S

(A4) $\qquad K_i E \subseteq K_i K_i E \qquad$ (*Axiom of Transparency*)

(A5) $\qquad -K_i E \subseteq K_i - K_i E \qquad$ (*Axiom of Wisdom*).

As before, I have labelled the restrictions on the assignment of propositions to events with the names of axioms of epistemic logic to which they correspond; here, however, they apply only to propositions concerning s.

Theorem 2. *A person has an i.p. for s in an e.m. μ of $\mathfrak{M}$ if and only if in μ her knowledge is closed w.r.t. s and strong w.r.t. s.*

Proof of Theorem 2. (*a*) Let S be any subset of Σ. For necessity we must show that (i) $K|S| = |S'|$ for some subset S' of Σ, (ii) $K|S| \subseteq KK|S|$, (iii) $-K|S| \subseteq K\text{–}K|S|$. Write J for the set $= \{j : S_j \subseteq S\}$ of the indices of the members of P that S completely includes. (i) From (2)–(4) and (7) we have, for any $s_0 \in S$,

$$|s_0| \subseteq \bigcap\nolimits_{-S_j}(K\text{–}|s|) \cap \bigcap\nolimits_{S_j}(-K\text{–}|s|). \tag{15}$$

Say $s_0 \in S_j$. Suppose first $j \in J$. From (15), $|s_0| \subseteq \bigcap_{-S_j} K\text{–}|s| = K\bigcap_{-S_j} -|s| = K|S_j| \subseteq K|S|$, by (A2) and Proposition 1 (since $|S_j| \subseteq |S|$ by nonepistemic assignment rules). Suppose on the other hand that $j \notin J$; then there is $s_1 \in S_j$ such that $|s_1| \subseteq -|S|$. But then $|s_0| \subseteq \bigcap_{S_j} -K\text{–}|s| \subseteq -K\text{–}|s_1| \subseteq -K|S|$ from (15) and Proposition 1. Since $\bigcap_\Sigma |s_0| = \Omega$ by the definition of $\mathfrak{M}$, $K|S| = \bigcup_J |S_j|$, and by the assignment rules

$$K|S| = |\bigcup\nolimits_J S_j| = |S'| \text{ say}. \tag{16}$$

(ii) By (16), $KK|S| = K|S'| = |\bigcup_H S_j|$ where $H = \{j : S_j \subseteq S'\}$, by the argument of (i). But then $H = J$ and $KK|S| = K|S|$. (iii) $K\text{–}|S| = |\bigcup_{-J} S_j| = K|S''|$ say, where $-J$ is the complement of J in I; whence, similarly, $K\text{–}K|S| = |\bigcup_H S_j|$ with $H = -J$, and so $K\text{–}K|S| = -K|S|$.

(b) Suppose knowledge is closed and strong w.r.t. s. By closedness, $KS \in \mathcal{S}$ for all S in $\mathcal{S}$, so writing K' for the restriction of K to $\mathcal{S}$, $\mu' = \langle \mathcal{S}, K' \rangle$ is an e.m. of $\mathfrak{M}$. Hence by Theorem 1 of Bacharach (1985) there is a partition Q of Σ such that in μ'

$$|\text{ip}\, Q| = \Omega \tag{17}$$

provided that, for all $S, S_1, S_2, \ldots$ in $\mathcal{S}$, (i) $KS \subseteq S$, (ii) $K(S_1 \cap S_2 \cap \ldots) = KS_1 \cap KS_2 \cap \ldots$, (iii) $KS \subseteq KKS$, (iv) $-KS \subseteq K\text{–}KS$. Now (i) holds by (A1), (ii) holds since (A2) implies $K(S_1 \cap S_2 \cap \ldots \cap S_n) = KS_1 \cap KS_2 \cap \ldots \cap KS_n$ for all n and Σ is finite, and (iii) and (iv) hold by hypothesis. Hence (17) holds in μ'; that is, for all s, $|s| \subseteq |\text{fix}\, Q(s)|$ in μ'. But $|\text{fix}\, Q(s)|$ in μ' is an intersection of sets of the forms $K'|S|$ and $-K'|S|$, $S \in \mathcal{S}$, and so identical to $|\text{fix}\, Q(s)|$ in μ. Thus for all s, $|s| \subseteq |\text{fix}\, Q(s)|$ in μ and (17) holds in μ. ∎

Remarks on Theorem 2. When is a state variable expistemically sufficient for itself? If s stands for a comprehensive specification of all relevant matters, it is so trivially, since the facts of s-knowledge are relevant matters. It is this case which underlies the Partition Theorem, and the Neo-Bayesian, globalist, concentration on this case helps to explain the unjust incrimination of i.p.'s by association with $S5$. But globality is not necessary: s is e.s.i. if it specifies states of affairs on which all knowledge, and so in particular s-knowledge, *supervenes*; that is, if there is necessarily a universal law which gives knowledge-states in terms of s. Such claims have been defended in case s is a complete *physical* description of the world (Kim 1978). However, this leaves s of astronomic dimension, and high dimension of the state variable militates against satisfaction of the strong axioms. In any case, it is of little help to the social scientist seeking a justification for the Partition Postulate for a topical variable t, for it only gives i.p.'s for s, from which we may not infer i.p.'s for t, even if t is a subvector of s. It is the variable we are modelling that we need to be e.s.i. One epistemic set-up in which this *is* so is one with which we are familiar—the simple experiment. For in an epistemic model canonical for e, a has an i.p. by Theorem 1, and her knowledge is therefore closed w.r.t. s by Proposition 3. Ironically, here, where we are assured of closedness, we have no need of it!

8. Remarks on strategies

Recall that standard accounts of how individuals may make decisions under uncertainty by means of strategies depend on their knowing their own (future) fix functions. By Proposition 4, someone knows her current fix function in the set-up described by μ if she has that fix function in μ. For this, by Theorems 1 and 2, it suffices that either she 'conduct a simple experiment' or her knowledge be closed and strong. This point may be generalized to deal with knowledge of future fix functions by treating the same person a at different times as two members of I, $a1$ and $a2$ say. For then a model μ which gives a basis of either kind for a's having an i.p. at time 2 allows us to conclude through Proposition 4 that $|K_{a1} K_{a2} \text{ip}_{a2} \phi| = \Omega$ in μ, in obvious notation.

The origins of information partitions lie in games in extensive form, where too they are instrumental in allowing rational choice to be defined in terms of strategies. I have suggested two ways of axiomatizing i.p.'s. It is of interest to ask whether either or both are plausible for the i.p.'s of players in the extensive form—and if both, which the more? Consider the situation of the player to move, and let s denote the current node. Note that s is e.s.i., because of two features in Von Neumann and Morgenstern's conception of a game: (a) the current node identifies the game's past (a game-tree has no loops); (b) a player's knowledge of the current node depends only on the *game's* past. It is at bottom (b) which, by ruling out mirrors, espionage and the wisdom of experience, makes a game an epistemically closed world. Theorem 2 now assures i.p.'s if players meet the full demands of $S5$.[18]

However, Von Neumann and Morgenstern's own explanation (1944, §9.1.5) of why players' information patterns are i.p.'s in effect proposes a simple experiment basis for them. The player to move is told, in the 'umpire's report', the values of certain 'personal functions' of the sequence of anterior choices, 'and no more'. 'This amount of information operates a subdivision of [the set of current nodes] into several disjunct subsets, corresponding to the various possible contents of [her] information'. Von Neumann and Morgenstern lack our means to do it, but clearly intend to make the assumption that a player knows her personal functions. Thus a player has an endowment of form e, and the proviso 'and no more' permits the conclusion that she conducts a simple experiment. Theorem 1 therefore provides formal grounds for Von Neumann and Morgenstern's explanation of i.p.'s; in addition, it shows that, in it, game-players only have to be T-rational.

The question whether players have i.p.'s arises elsewhere in game theory. May we so represent their uncertainty about the n-tuple s of strategies? Aumann (1987) assumes i.p.'s with respect to a state variable describing 'all parameters that may be the object of uncertainty' for a player. However, we have seen that even if this assumption could be justified, say by appeal to the Partition Theorem, it would not give i.p.'s for the 'topical' variable s. Moreover, since i.p.'s do not 'project outwards', the i.p.'s over courses of play for which our theorems provide grounds do not imply i.p.'s for a comprehensive parameter-vector including s.

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[18] Moreover, in this case some impoverishment of the umpire's report is in general possible, as the example of Fig. 3 illustrates.

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